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A characteristic initial-value problem approach to the formation of inhomogeneities in the early universe

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Abstract. Starting from a small perturbation in the initial data for a Robertson–Walker universe, it is shown that, in a small region of space-time, it is possible for large fluctuations to develop in both the metric and physical variables describing the universe. The initial data are specified on a characteristic hypersurface in a region of perfect fluid filled space–time and the method of analysis of the field equations is based on the techniques introduced by Bondi, van der Burg and Metzner.

1. Introduction

One of the problems facing cosmologists is to explain how galaxies condensed in a universe which, on the large scale, is homogeneous and isotropic. There appear to be two possibilities: either matter after the initial big bang was in a highly chaotic state, which has since become homogeneous and isotropic with just enough random fluctuation at the appropriate time to form galaxies (see, for example, Misner *et al* (1973)), or else the situation immediately after matter had formed was homogeneous and isotropic, and galaxies have arisen as a result of small perturbations from this state. In this paper the latter possibility will be considered in more detail.

It has been thought that if suitable initial data were built into the description of the universe after the big bang, then perturbations would develop and carry on developing. However, it has been difficult to demonstrate how this happens to a universe based on a standard Robertson–Walker cosmological model by solving an initial-value problem with data specified on a space-like hypersurface (Hawking 1966, Weinberg 1972). I have attempted here to show how these perturbations may begin to develop, by considering an initial-value problem for a small region of space–time with data specified on a null hypersurface. The more difficult question of relating the formal solution of the initial-value problems to galaxy formation is not considered here. The approach to solving characteristic initial-value problems in general relativity used here was instigated by Bondi *et al* (1962) and adapted towards the problem under discussion in two previous papers, Chellone and Williams (1973) and Chellone (1976), referred to as (I) and (II).

In (II) a formal solution to the characteristic initial-value problem for a perfect fluid filled Robertson–Walker universe was generated for a region near to an origin world line within the fluid. This restriction is necessary, as the method of solution of the problem requires all metric and physical variables to be expanded in positive powers of

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a radial parameter. The initial data can then be shown to be composed of coefficients in certain of the power series expansions together with an equation of state which gives the density as a function of the presssure.

In the present paper small perturbations in the characteristic initial data for a Robertson–Walker model are built into the analysis. The result of doing this is that, in a neighbourhood of an origin world line, it is possible to obtain significant variations in the non-data variables, and that the perturbations in both data and non-data variables either maintain themselves or increase with increasing time.

2. The Robertson–Walker universe

The metric describing a homogeneous, isotropic space-time is (Weinberg 1972)

$$ds^2 = dt'^2 - R(t')^2\left(\frac{dr'^2}{1-kr'^2} + d\theta'^2 + \sin^2\theta'\, d\phi'^2\right)$$

where the coordinates are (t', r', θ', ϕ'), $R(t')$ is the 'scale factor' for the universe and k, the curvature, is -1, 0 or $+1$. The Bondi form of metric describing a general axially and azimuth reflection symmetric space–time is (Bondi *et al* 1962)

$$ds = (V\,e^{2\beta}/r - U^2r^2\,e^{2\gamma})\,du^2 + 2\,e^{2\beta}\,du\,dr$$
$$+2Ur^2\,e^{2\gamma}\,du\,d\theta - r^2(e^{2\gamma}\,d\theta^2 + e^{-2\gamma}\sin^2\theta\,d\phi^2),$$

where U, V, β and γ are functions of the coordinates u, r and θ. The full coordinate system used is (u, r, θ, ϕ), which are numbered according to the scheme $(0, 1, 2, 3)$. Incoming radiation will be excluded from the analysis.

Transforming the Robertson–Walker metric into the Bondi form results in (II)

$$g_{00} = 1 - (rR^{-2}\,dR/da)^2R/(1-kr^2R^{-2}) = V\,e^{2\beta}/r,$$
$$g_{01} = df/dr - (rR^{-2}dR/da)[R^{-1} - rR^{-2}(dR/da)(df/dr)]R/(1-kr^{-2}R^{-2}) = e^{2\beta},$$
$$g_{22} = -r^2, \qquad g_{33} = -r^2\sin^2\theta,$$

where

$$t' = u + f(r) = a(u, r), \qquad r' = r/R(a),$$
$$\theta' = \theta, \qquad \phi' = \phi,$$

and U and γ are zero owing to the spherical symmetry of the Robertson–Walker metric.

With the assumption that space–time is filled with a perfect fluid which has energy–momentum tensor

$$T_{\mu\nu} = (p+\rho)v_\mu v_\nu - pg_{\mu\nu}$$

where p is the pressure, ρ the density and v^μ the velocity vector for the fluid, the field equations and the conservation condition $T^{\mu\nu}{}_{;\nu} = 0$ lead to a solution of the characteristic initial-value problem in the case where all variables are expanded in positive powers of r. (For details see (I) and (II).)

The non-zero initial data for the Robertson–Walker metric can be chosen to be p, v_1 and an equation of state of the form $\rho = \rho(p)$ together with an arbitrary function of

r-integration, $\mathring{\beta}$. The data variables γ and v_2 are zero. So, specifying the leading coefficients of the non-zero data variables on a particular null hypersurface u = constant together with $\mathring{\beta}(u)$ enables the leading terms of all the other physical and metric variables to be obtained on u = constant. The values of these variables on future null hypersurfaces can be found once the time development equations for the data variables have been considered.

3. Perturbations from the Robertson–Walker metric

To generalise the metric away from exact spherical symmetry, a small perturbation is introduced into the data variables of the symmetric initial-value problem. This is done by introducing a small, but non-zero, parameter ε. Data variables independent of ε are those non-zero variables from the exact Robertson–Walker solution which, when expanded in positive powers of r, have the form

$$p=\mathring{p}+rD+\mathrm{O}(r^2), \qquad v_1=1+\mathrm{O}(r^2).$$

The first coefficient of the ρ expansion is determined by the equation of state.

If all the other coefficients in the data variables for a general Bondi perfect fluid space–time represent the small perturbation, then referring to (I)

$$\gamma=r^2 f\sin^2\theta+r^3(g\sin^2\theta+h\sin^2\theta\cos\theta)+\mathrm{O}(r^4),$$
$$p=\mathring{p}+r(D+E\cos\theta)+\mathrm{O}(r^2),$$
$$v_1=(1+B^2)^{1/2}-B\cos\theta+r(a+b\cos\theta+d\cos^2\theta)+\mathrm{O}(r^2),$$
$$v_2=rB\sin\theta+r^2(k\sin\theta+l\sin\theta\cos\theta)+\mathrm{O}(r^3),$$

where f, g, h, E, B, a, b, k, l and d are of order ε. The forms of v_1 and v_2 arise from continuity conditions on $r=0$ (see (I) for details). v_0 can be obtained from the normality condition $v^\mu v_\mu=1$. Not all of the coefficients are independent, and if the expansion for ρ is

$$\rho=\mathring{\rho}+r(F+G\cos\theta)+\mathrm{O}(r^2),$$

the following relationships hold:

$$\mathring{\rho}=\mathring{\rho}(\mathring{p}) \qquad \text{(equation of state)},$$
$$l=-d,$$
$$F=D\rho', \qquad \rho'=\mathrm{d}\rho/\mathrm{d}p \text{ evaluated at } r=0,$$
$$E=-FB(1+B^2)^{-1/2}/\rho'+(\mathring{p}+\mathring{\rho})[2B(a+d)+b(1+2B^2)(1+B^2)^{-1/2}],$$
$$G=-B^{-1}(1+B^2)^{1/2}F+(\mathring{p}+\mathring{\rho})[(3a+d)B^{-1}+4B(a+d)+4b(1+B^2)^{1/2}],$$
$$k=-B(1+B^2)^{1/2}(a+d)-b-bB^2.$$

These relationships are consistent with the ε dependence given above, while the order of magnitude behaviour of the coefficients in ρ is

$$F=\mathrm{O}(1), \qquad G=\mathrm{O}(1/\varepsilon).$$

It is this last result that enables large perturbations to occur.

When the first few terms of the non-data metric variables β, U and V have been computed, the results are:

$$\beta = \overset{\circ}{\beta}(1) + r^2\overset{2}{\beta}(\varepsilon) + r^3\overset{3}{\beta}(1/\varepsilon) + \cdots,$$

$$U = r\overset{1}{U}(\varepsilon) + r^2\overset{2}{U}(1/\varepsilon) + \ldots, \qquad V = r + r^3\overset{3}{V}(1) + \ldots,$$

where the order of magnitude behaviour in terms of ε is explicitly shown for each coefficient in the r-expansion. $\overset{\circ}{\beta}$ is the arbitrary function of r-integration previously introduced.

Hence, on an initial null hypersurface, the result of introducing a small perturbation away from a Robertson–Walker metric is to produce on the hypersurface perturbations of the same order in the first few terms of p, v_1 and v_2, but to produce large perturbations in some coefficients of β, U and V. These perturbations, although formally of order $1/\varepsilon$, are not arbitrarily large as, in all cases, the terms concerned are multiplied by, at least, a factor of r; which, as the expansion is valid only in a neighbourhood of $r = 0$, will be sufficient to prevent infinitely large perturbations.

The next question is whether or not these large perturbations maintain themselves with increasing time. To answer this, the time development of the data variables must be considered. An analysis of these equations (equations (4.10)–(4.13) of (I)) leads to

$$\gamma_{,0}X(1) = Y(1/\varepsilon),$$

$$p_{,0}A(1) + v_{1,0}B(1/\varepsilon) + v_{2,0}C(1) = D(1/\varepsilon),$$

$$p_{,0}E(1) + v_{1,0}F(1/\varepsilon) = G(1/\varepsilon),$$

$$p_{,0}H(\varepsilon) + v_{1,0}I(1) + v_{2,0}J(1/\varepsilon) = K(1/\varepsilon),$$

where the capital letters denote functions known on the original hypersurface of the indicated order of ε and a comma denotes a partial derivative.

An order of magnitude calculation for the time derivatives leads, in general, to the following behaviour:

$$\gamma_{,0} = \mathrm{O}(1/\varepsilon), \qquad p_{,0} = \mathrm{O}(1/\varepsilon^2),$$

$$v_{1,0} = \mathrm{O}(1/\varepsilon), \qquad v_{2,0} = \mathrm{O}(1),$$

which shows that the perturbations of all the data variables will increase as time advances.

The formal solution to the characteristic initial-value problem can now be given in terms of these data variables, the arbitrary function $\overset{\circ}{\beta}(u)$ and the equation of state $\rho = \rho(p)$.

4. Conclusion

An analysis of a perturbed Robertson–Walker metric by means of a characteristic initial-value problem shows that, in a neighbourhood of an origin world line, small perturbations away from exact spherical symmetry are capable of producing and sustaining large variations in some of the metric and physical variables. The implication of this is that it is possible, at least in a small region, for large inhomogeneities to develop in an initially smoothed out and spherically symmetric universe.

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